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ON AN INVARIANT RELATION OF DYNAMICAL SYSTEMS

BY

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ON AN INVARIANT RELATION OF DYNAMICAL SYSTEMS

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§ 1. *Introduction*

Wintner[1] has given an invariant relation for dynamical systems for which,

$$H = T - U = \tfrac{1}{2} g^{ij} p_i p_j - U \tag{1}$$

where g^{ij} is of degree α in the q_r's, and U of degree β, both being homogeneous functions.

Also

$$T = \tfrac{1}{2} g_{ij} \dot{q}_i \dot{q}_j \tag{2}$$

with g_{ij} being homogeneous of degree $-\alpha$. The invariant relation in question is

$$\frac{d}{dt}(g_{ij} q_i \dot{q}_j) = (\beta - \alpha + 2) U \tag{3}$$

which is valid along those solutions of the dynamical system running in the sub-space $h = 0$. The method used by Wintner is an application of Darboux's linear transformation.

The relation (3) has also been derived by Pars[2] for the case $h \neq 0$ by multiplying the left-hand members of Lagrange's rth equation by q_r and summing up for $r = 1$ to n. The relation he obtains is

$$\begin{aligned} \frac{d}{dt}(a_{ij} q_i \dot{q}_j) &= \beta U - (\alpha - 2) T \\ &= (\beta - \alpha + 2) U - (\alpha - 2) h \end{aligned} \tag{4}$$

I give here two proofs of Wintner's relations, one direct and extremely simple, and the other which brings out the relation of Wintner's result to the theory of integrals linear in the momenta. This latter view-point enables us to find another invariant relation which is an extension of Wintner's result.

§ 2. *Preliminary remarks and direct proof*

It might be noticed in passing that Wintner's relation is an extension of Lagrange's identity for the n-body problem,[3] *viz.*,

$$\frac{d^2 R^2}{dt^2} = 2(U - 2K) \qquad (K = U - T)$$

where $R^2 = \frac{1}{2M} \Sigma m_i m_j r_{ij}^2$. In fact, we can write

$$\frac{dR^2}{dt} = \frac{1}{M} \Sigma m_i m_j r_{ij} \dot{r}_{ij}$$

which can, by a suitable choice of co-ordinates, be reduced to $\Sigma a_{ij} q_i \dot{q}_j$, and putting $\beta = -1$, and $\alpha = 0$ in Wintner's relation (4) we get Lagrange's identity.

To prove (4) directly we write

$$\Sigma p_r \dot{q}_r - L = H$$

i.e.

$$\frac{d}{dt} (\Sigma p_r q_r) - \Sigma q_r \dot{p}_r - L = H$$

$$\frac{d}{dt} (\Sigma p_r q_r) = \Sigma q_r \frac{\partial H}{\partial q_r} + H + L = \Sigma q_r \frac{\partial H}{\partial q_r} + 2T$$

and in virtue of the homogeneity of T and U the result follows at once.

§ 3. *Theory of integrals linear in the momenta*

The method of Pars is a special case of Birkhoff's method of multipliers[4] for the determination of integrals linear in the momenta. Let us now investigate the question of

$$\phi = \Sigma q_i p_i + S(q_1, q_2, \cdots q_n) \tag{5}$$

or

$$\phi = \Sigma a_{ij} q_i \dot{q}_j + S$$

being an integral of the canonical system

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}; \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \tag{6}$$

Although relation (4) *is not a consequence of this condition*, it follows as a natural consequence of the method of procedure.

Corresponding to ϕ we can define an infinitesimal contact transformation

$$\left.\begin{aligned} \delta q_1 &= \epsilon q_1 \\ \delta p_1 &= -\epsilon p_1 \end{aligned}\right\} \tag{7}$$

and $-\frac{d\phi}{dt} = -(\phi, H) =$ symbol of the transformation (7) and hence equal to δH. Hence

$$\frac{d\phi}{dt} = -\delta H = \delta U - \delta T \tag{8}$$

Since $T = \frac{1}{2} g^{ij} p_i p_j$, the transformation (7) gives

$$\delta U = \beta U, \text{ and } \delta T = \alpha T - 2T$$

which proves immediately Wintner's relation. We might repeat this method

and consider

$$\left(\phi, \frac{d\phi}{dt}\right) = -\,\delta\left(\frac{d\phi}{dt}\right) = \delta\,(\alpha - 2)\,\mathrm{T} - \delta\beta\mathrm{U}$$
$$= (\alpha - 2)^2\,\mathrm{T} - \beta^2\mathrm{U} \tag{9}$$

which is also an invariant relation analogous to (4) and a sort of generalisation of it.

Coming now to the question of (5) being an integral we might mention two cases :—

(i) $\alpha = 2$, where we can write $\mathrm{S} = \beta\tau$ with $\tau = \int^{t} \mathrm{U}dt$ giving

$$\phi = \Sigma\, p_i\, q_i + \beta\tau$$

an integral linear in the momenta.

(ii) In the general case of α and β having any values we can have

$$\phi = \Sigma\, p_i\, q_i + (\beta - \alpha + 2)\,\tau$$

as a conditional linear integral,[5] *i.e.* holding only for the specified value $h = 0$.

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3. Birkhoff .. *Dynamical Systems*, pp. 265 and 291.
4. ,, .. *Ibid.*, p. 41, § 6.
5. ,, .. *Ibid.*, p. 45, § 8.

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